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THE CONCEPT OF CONVERGENCE FOR 2-DIMENSIONAL SUBSPACES SEQUENCE IN NORMED SPACES

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ABSTRACT. In this paper, we present a concept of convergence of sequence, especially, of 2-dimensional subspaces of normed spaces. The properties of the concept are established. As sequences of our definition in an inner product space, we also obtain the continuity property of the angle between two 2-dimensional subspaces of inner product spaces.

Key words and phrases: Convergence of sequence; Inner product spaces; Normed spaces.

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1. INTRODUCTION

In an inner product space $(X, \langle\cdot,\cdot\rangle)$, the angle $A(x, y)$ between two nonzero vectors x and y in X is usually given by

$$A(x, y) = \arccos \frac{\langle x, y\rangle}{\|x\|\|y\|}$$

where $\|x\| := \langle x, y\rangle^{1/2}$ denotes the induced norm in X. One may observe that the angle $A(\cdot,\cdot)$ in X satisfies the following basic properties (see [7]):

(1) Parallelism: $A(x, y) = 0$ if and only if x and y are of the same direction; $A(x, y) = \pi$ if and only if x and y are of opposite direction.
(2) Symmetry: $A(x, y) = A(y, x)$ for every x, y in X.
(3) Homogeneity: $A(ax, by) = A(x, y)$ if $ab > 0$; $A(ax, by) = \pi - A(x, y)$ if $ab < 0$.
(4) Continuity: If $x_n \to x$ dan $y_n \to y$, then $A(x_n, y_n) \to A(x, y)$.

Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space of dimension 2 or higher (may be infinite), and $U = span\{u_1, u_2\}$ and $V = span\{v_1, v_2\}$ be 2-dimensional subspaces of X, where $\{u_1, u_2\}$ and $\{v_1, v_2\}$ are orthonormal. Then, as is suggested by Gunawan, Neswan and Setya-Budhi in [8], one can define the angle $\theta(U, V)$ be the angle between U and V, given by

$$\cos^2 \theta(U, V) = \det \theta(M^T M)$$

where $M = [\langle u_i, v_k\rangle]^T$ being a 2×2 matrix. It is can be seen that $\theta(U, V)$ satisfies the basic properties of angle, i.e.

(1) $\theta(U, V) = 0$ if and only if $U = V$ (parallelism property)
(2) $\theta(U, V) = \theta(V, U)$ for every $U, V \subset X$ (symmetry property)
(3) $\theta(U, V)$ choice of basis independent (homogeneity property).

The focus in this article is what is the continuity property of $\theta(\cdot,\cdot)$. In the other words, if $U_n \to U$ and $V_n \to V$ will it be valid for $\theta(U_n, V_n) \to \theta(U, V)$. This problem can be solved if we have a convergence definition for sequences of 2-dimensional subspaces. Therefore, we will first construct the convergence concept.

As foundation in constructing convergence concept of 2-dimensional subspaces, in 2018, Manuharawati et al. [11], have introduced a more geometrical definition of the convergence of a one-dimensional sequence of a one-dimensional subspace of a normed space and explained its connection with angle between two 1-dimensional subspace of a innerproduct space. Their definition of the convergence of a sequence of 1-dimensional subspace, however, is based on a definition convergence of a sequence on vector which we found a gap which we have generalized it to 2-dimensional subspace. The purpose of this note is to fix their definition and at the same time to explain the relationship between convergence of sequence with the angle between two 2-dimensional subspaces of real inner product space. In this paper, we construct the concept of convergence for sequence of two dimensional vector subspaces of a vector space, discuss the basic properties, and use the concept to prove the continuity of angle between two vector subspaces. In order to achieve that, we need a metric in Grasmanian.

2. RESULT

Let $(X, \|\cdot\|)$ be a real normed space with $dim(X) > 2$. Given a sequence of 2-dimensional subspaces (U_n) and a subspaces U of X. We wish to have a definition of the convergence of a sequence of (U_n) to U as $n \to \infty$. To define the limit of a sequence of 2-dimensional subspaces, we have to revisit the grassmannian space ([9]). The Grassmannian is a fundamental object of study across various subdisciplines of modern geometry. The reason why we are interested in the Grassmannian is because we eventually wish to define a metric, on the Grassmannian, so that we can define convergence of sequence and many other notions on this space.

Definition 2.1. Let $(X, \|\cdot\|)$ be a real normed space with $dim(X) > k$. The Grassmannian $Gr(k, X)$ is a set of all k-dimensional linear subspaces of X.

$$Gr(k, X) = \{U \subset X \,:\, U \,\text{is a}\, k - \text{dimensional subspace of}\, X\}$$

For example, the Grassmannian $Gr(1, X)$ is the space of lines through the origin in X, so it is the same as the projective space of one dimension lower than X ([1]). The Grassmannian $Gr(2, X)$ is the space of planes through the origin in X.

Theorem 2.1. *If d_* be a real valued function on $Gr(2, X) \times Gr(2, X)$ with*

$$\begin{aligned} d_*(U, V) &= \max\Big(\sup_{\substack{u \in U \\ \|u\|=1}} \inf_{\substack{v \in V \\ \|v\|=1}} \|u - v\|, \sup_{\substack{v \in V \\ \|v\|=1}} \inf_{\substack{u \in U \\ \|u\|=1}} \|v - u\|\Big) \\ &= \max\Big(\sup_{u \in U} \inf_{v \in V} \Big\|\frac{u}{\|u\|} - \frac{v}{\|v\|}\Big\|, \sup_{v \in V} \inf_{u \in U} \Big\|\frac{v}{\|v\|} - \frac{u}{\|u\|}\Big\|\Big), \quad u, v \neq 0, \end{aligned}$$

then $(Gr(2, X), d_*)$ is a metric space.

Proof. Let U and V are two 2-dimensional subspaces of a normed space X.
(i) We have known that $\Big\|\frac{u}{\|u\|} - \frac{v}{\|v\|}\Big\| \geq 0$ for all $u, v \in X - \{0\}$.
Thus, $\inf_{v \in V} \Big\|\frac{u}{\|u\|} - \frac{v}{\|v\|}\Big\| \geq 0$ and $\inf_{u \in U} \Big\|\frac{v}{\|v\|} - \frac{u}{\|u\|}\Big\| \geq 0$.
Consequently, $\sup_{u \in U} \inf_{v \in V} \Big\|\frac{u}{\|u\|} - \frac{v}{\|v\|}\Big\| \geq 0$ and $\sup_{v \in V} \inf_{u \in U} \Big\|\frac{v}{\|v\|} - \frac{u}{\|u\|}\Big\| \geq 0$.
So, $d_*(U, V) \geq 0$.
If $U = V$, then $Gr(2, X) \times Gr(2, X)$ with

$$\begin{aligned} d_*(U, V) &= \max\Big(\sup_{\substack{u \in U \\ \|u\|=1}} \inf_{\substack{v \in V \\ \|v\|=1}} \|u - v\|, \sup_{\substack{v \in V \\ \|v\|=1}} \inf_{\substack{u \in U \\ \|u\|=1}} \|v - u\|\Big) \\ &= \max\Big(\sup_{u \in U} \inf_{v \in V} \Big\|\frac{u}{\|u\|} - \frac{v}{\|v\|}\Big\|, \sup_{v \in V} \inf_{u \in U} \Big\|\frac{v}{\|v\|} - \frac{u}{\|u\|}\Big\|\Big) = 0. \end{aligned}$$

If $d_*(U, V) = 0$, then

$$\sup_{u \in U} \inf_{v \in V} \Big\|\frac{u}{\|u\|} - \frac{v}{\|v\|}\Big\| = \sup_{v \in V} \inf_{u \in U} \Big\|\frac{v}{\|v\|} - \frac{u}{\|u\|}\Big\| = 0$$

Consequently, For every $u \in U$, we have

$$\inf_{v \in V} \Big\|\frac{u}{\|u\|} - \frac{v}{\|v\|}\Big\| = 0$$

and for every $v \in V$, we also have

$$\inf_{u \in U} \Big\|\frac{v}{\|v\|} - \frac{u}{\|u\|}\Big\| = 0.$$

Because U and V are closed (see Theorem 2.4-3 in [10], every finite dimensional subspace of normed spaces is closed), then there are $v_o \in V$ and $u_o \in U$ such that

$$\Big\|\frac{u}{\|u\|} - \frac{v_0}{\|v_0\|}\Big\| = \Big\|\frac{v}{\|v\|} - \frac{u_0}{\|u_0\|}\Big\| = 0.$$

Therefore there are $\alpha, \beta \in \mathbb{R}$ such that $u = \alpha v_0$ and $v = \beta u_0$. That means, for every $u \in U$ and $v \in V$ then $u \in V$ and $v \in U$. So $U = V$.

(ii) From definition of d_*, we have $d_*(U,V) = d_*(V,U)$ for every U and V in $Gr(2,X)$.
(iii) We can observe that

$$\begin{aligned}
\sup_{u\in U}\inf_{v\in V}\left\|\frac{u}{\|u\|}-\frac{v}{\|v\|}\right\| &= \sup_{u\in U}\inf_{w\in W}\inf_{v\in V}\left\|\frac{u}{\|u\|}-\frac{v}{\|v\|}\right\| \\
&= \sup_{u\in U}\inf_{w\in W}\inf_{v\in V}\left\|\frac{u}{\|u\|}-\frac{w}{\|w\|}+\frac{w}{\|w\|}-\frac{v}{\|v\|}\right\| \\
&\le \sup_{u\in U}\inf_{w\in W}\inf_{v\in V}\left\|\frac{u}{\|u\|}-\frac{w}{\|w\|}\right\|+\sup_{u\in U}\inf_{w\in W}\inf_{v\in V}\left\|\frac{w}{\|w\|}-\frac{v}{\|v\|}\right\| \\
&= \sup_{u\in U}\inf_{w\in W}\left\|\frac{u}{\|u\|}-\frac{w}{\|w\|}\right\|+\inf_{w\in W}\inf_{v\in V}\left\|\frac{w}{\|w\|}-\frac{v}{\|v\|}\right\| \\
&\le \sup_{u\in U}\inf_{w\in W}\left\|\frac{u}{\|u\|}-\frac{w}{\|w\|}\right\|+\sup_{w\in W}\inf_{v\in V}\left\|\frac{w}{\|w\|}-\frac{v}{\|v\|}\right\| .....(1)
\end{aligned}$$

and

$$\begin{aligned}
\sup_{v\in V}\inf_{u\in U}\left\|\frac{v}{\|v\|}-\frac{u}{\|u\|}\right\| &= \sup_{v\in V}\inf_{w\in W}\inf_{u\in U}\left\|\frac{v}{\|v\|}-\frac{u}{\|u\|}\right\| \\
&= \sup_{v\in V}\inf_{w\in W}\inf_{u\in U}\left\|\frac{v}{\|v\|}-\frac{w}{\|w\|}+\frac{w}{\|w\|}-\frac{u}{\|u\|}\right\| \\
&\le \sup_{v\in V}\inf_{w\in W}\inf_{u\in U}\left\|\frac{v}{\|v\|}-\frac{w}{\|w\|}\right\|+\sup_{v\in V}\inf_{w\in W}\inf_{u\in U}\left\|\frac{w}{\|w\|}-\frac{u}{\|u\|}\right\| \\
&= \sup_{v\in V}\inf_{w\in W}\left\|\frac{v}{\|v\|}-\frac{w}{\|w\|}\right\|+\inf_{w\in W}\inf_{u\in U}\left\|\frac{w}{\|w\|}-\frac{u}{\|u\|}\right\| \\
&\le \sup_{v\in V}\inf_{w\in W}\left\|\frac{v}{\|v\|}-\frac{w}{\|w\|}\right\|+\sup_{w\in W}\inf_{u\in U}\left\|\frac{w}{\|w\|}-\frac{u}{\|u\|}\right\| .....(2)
\end{aligned}$$

From (1) and (2), we get

$$d_*(U,V) \le d_*(U,W) + d_*(W,V).$$

∎

3. MAIN RESULT

By referring the metric spaces constructed in section 2, we therefore construct 2-dimensional vector subspace sequence convergence and its properties. Let $(X, \|\cdot\|)$ be a real normed space with $dim(X) > 2$, (U_n) be a sequence of 2-dimensional subspaces in X, and U be a 2-dimensional subspace of X. A sequence (U_n) is said to be converge to U if and only if

$$d_*(U_n, U) \to 0$$

as $n \to \infty$.

Definition 3.1. Let (U_n) be a sequence of 2-dimensional subspaces in X and U be a 2-dimensional subspace in X. A sequence (U_n) is said to be converge to U if

$$\max\left(\sup_{u_n\in U_n}\inf_{u\in U}\left\|\frac{u_n}{\|u_n\|}-\frac{u}{\|u\|}\right\|, \sup_{u\in U}\inf_{u_n\in U_n}\left\|\frac{u}{\|u\|}-\frac{u_n}{\|u_n\|}\right\|\right) \to 0$$

as $n \to \infty$.

We shall use the following phrase and notation interchangeably: (U_n) converges to U or $U_n \to U$ as $n \to \infty$ or the limit of (U_n) exists and equals U.

Given $U_n = span\{u_{1_n}, u_{2_n}\}$ and $U = span\{u_1, u_2\}$, then we get

$$\begin{aligned} d_*(U_n, U) &= \max\Big(\sup_{u_n\in U_n}\inf_{u\in U}\Big\|\frac{u_n}{\|u_n\|} - \frac{u}{\|u\|}\Big\|, \sup_{u\in U}\inf_{u_n\in U_n}\Big\|\frac{u}{\|u\|} - \frac{u_n}{\|u_n\|}\Big\|\Big) \\ &= \max\Big(\sup_{a,b\in\mathbb{R}}\inf_{c,d\in\mathbb{R}}\Big\|\frac{au_{1_n}+bu_{2_n}}{\|au_{1_n}+bu_{2_n}\|} - \frac{cu_1+du_2}{\|cu_1+du_2\|}\Big\|, \\ &\qquad \sup_{c,d\in\mathbb{R}}\inf_{a,b\in\mathbb{R}}\Big\|\frac{cu_1+du_2}{\|cu_1+du_2\|} - \frac{au_{1_n}+bu_{2_n}}{\|au_{1_n}+bu_{2_n}\|}\Big\|\Big). \end{aligned}$$

So, we have the following reformulation of Definition 3.1.

Definition 3.2. Let (U_n) be a sequence of 2-dimensional subspaces in X with $U_n = span\{u_{1_n}, u_{2_n}\}$ and $U = span\{u_1, u_2\}$ be a 2-dimensional subspace in X. A sequence (U_n) is said to be converge to U if

$$\max\Big(\sup_{a,b\in\mathbb{R}}\inf_{c,d\in\mathbb{R}}\Big\|\frac{au_{1_n}+bu_{2_n}}{\|au_{1_n}+bu_{2_n}\|} - \frac{cu_1+du_2}{\|cu_1+du_2\|}\Big\|, \sup_{c,d\in\mathbb{R}}\inf_{a,b\in\mathbb{R}}\Big\|\frac{cu_1+du_2}{\|cu_1+du_2\|} - \frac{au_{1_n}+bu_{2_n}}{\|au_{1_n}+bu_{2_n}\|}\Big\|\Big) \to 0$$

as $n \to \infty$.

If a sequence has a limit, we say that the sequence is convergent; if it has no limit, we say that the sequence is divergent.We will sometimes use the symbolism $U_n \to U$, which indicates the intuitive idea that the lines U_n "approach" the line U as $n \to \infty$. The following theorem convinces us that our definition make sense.

Theorem 3.1. *The Definition 3.2 satisfy homogeneity property. It means, the definition is independent on the choice of bases for U and U_n for every $n \in \mathbb{N}$.*

Proof. First note that the value of

$$\max\Big(\sup_{a,b\in\mathbb{R}}\inf_{c,d\in\mathbb{R}}\Big\|\frac{au_{1_n}+bu_{2_n}}{\|au_{1_n}+bu_{2_n}\|} - \frac{cu_1+du_2}{\|cu_1+du_2\|}\Big\|, \sup_{c,d\in\mathbb{R}}\inf_{a,b\in\mathbb{R}}\Big\|\frac{cu_1+du_2}{\|cu_1+du_2\|} - \frac{au_{1_n}+bu_{2_n}}{\|au_{1_n}+bu_{2_n}\|}\Big\|\Big)$$

is independent on the choice of basis for U_n and U for every $n \in N$. Further, since the value is also invariant under any change of basis for U_n and U. Indeed, the ratio is unchanged if we (a) swap u_{1_n} and u_{2_n}, (b) swap u_1 and u_2, (c) replace u_{1_n} by $u_{1_n} + \alpha u_{2_n}$, (d) replace u_1 by $u_1 + \alpha u_2$, (e) replace u_{1_n} by αu_{1_n}, (f) replace u_{2_n} by αu_{2_n}, (g) replace u_1 by αu_1, or (h) replace u_2 by αu_2 with $\alpha \neq 0$. ∎

Theorem 3.2. *If $U_n = span\{u_1, u_2\} = U$ then $U_n \to U$ as $n \to \infty$.*

Proof.

$$\begin{aligned} &\max\Big(\sup_{a,b\in\mathbb{R}}\inf_{c,d\in\mathbb{R}}\Big\|\frac{au_{1_n}+bu_{2_n}}{\|au_{1_n}+bu_{2_n}\|} - \frac{cu_1+du_2}{\|cu_1+du_2\|}\Big\|, \sup_{c,d\in\mathbb{R}}\inf_{a,b\in\mathbb{R}}\Big\|\frac{cu_1+du_2}{\|cu_1+du_2\|} - \frac{au_{1_n}+bu_{2_n}}{\|au_{1_n}+bu_{2_n}\|}\Big\|\Big) \\ &= \max\Big(\sup_{a,b\in\mathbb{R}}\inf_{c,d\in\mathbb{R}}\Big\|\frac{au_1+bu_2}{\|au_1+bu_2\|} - \frac{cu_1+du_2}{\|cu_1+du_2\|}\Big\|, \sup_{c,d\in\mathbb{R}}\inf_{a,b\in\mathbb{R}}\Big\|\frac{cu_1+du_2}{\|cu_1+du_2\|} - \frac{au_1+bu_2}{\|au_1+bu_2\|}\Big\|\Big) \\ &\quad = \max\Big(\sup_{a,b\in\mathbb{R}}\Big\|\frac{au_1+bu_2}{\|au_1+bu_2\|} - \frac{au_1+bu_2}{\|au_1+bu_2\|}\Big\|, \sup_{c,d\in\mathbb{R}}\Big\|\frac{cu_1+du_2}{\|cu_1+du_2\|} - \frac{cu_1+du_2}{\|cu_1+du_2\|}\Big\|\Big) \\ &= 0 \to 0 \end{aligned}$$

as $n \to \infty$. ∎

Theorem 3.3. (**Uniqueness of Limit**) *Let $U_n = span\{u_{1_n}, u_{2_n}\}$, $U = span\{u_1, u_2\}$, and $V = span\{v_1, v_2\}$. If $U_n \to U$ and $U_n \to V$ as $n \to \infty$, then $U = V$.*

Proof. Let $U_n \to U$ as $n \to \infty$. By Definition 3.2, we have

$$d_*(U_n, U) = \max\Big(\sup_{a,b\in\mathbb{R}} \inf_{c,d\in\mathbb{R}} \Big\| \frac{au_{1_n} + bu_{2_n}}{\|au_{1_n} + bu_{2_n}\|} - \frac{cu_1 + du_2}{\|cu_1 + du_2\|} \Big\|,$$

$$\sup_{c,d\in\mathbb{R}} \inf_{a,b\in\mathbb{R}} \Big\| \frac{cu_1 + du_2}{\|cu_1 + du_2\|} - \frac{au_{1_n} + bu_{2_n}}{\|au_{1_n} + bu_{2_n}\|} \Big\|\Big) \to 0$$

as $n \to \infty$. Since $U_n \to V$ as $n \to \infty$ then we also have

$$d_*(U_n, V) = \max\Big(\sup_{a,b\in\mathbb{R}} \inf_{c,d\in\mathbb{R}} \Big\| \frac{au_{1_n} + bu_{2_n}}{\|au_{1_n} + bu_{2_n}\|} - \frac{cv_1 + dv_2}{\|cv_1 + dv_2\|} \Big\|,$$

$$\sup_{c,d\in\mathbb{R}} \inf_{a,b\in\mathbb{R}} \Big\| \frac{cv_1 + dv_2}{\|cv_1 + dv_2\|} - \frac{au_{1_n} + bu_{2_n}}{\|au_{1_n} + bu_{2_n}\|} \Big\|\Big) \to 0$$

as $n \to \infty$.
By using Triangle Inequality of d_*, we get

$$d_*(U, V) \le d_*(U, U_n) + d_*(U_n, V) \to 0$$

as $n \to \infty$. So, $d_*(U, V) = 0$. Consequently, we have $U = V$. ∎

Theorem 3.4. *Let $U_n = span\{u_{1_n}, u_{2_n}\}$ and $U = span\{u_1, u_2\}$. If $U_n \to U$ then*

$$\max\Big(\max_{i\in\{1,2\}} \inf_{c,d\in\mathbb{R}} \Big\| \frac{u_{i_n}}{\|u_{i_n}\|} - \frac{cu_1 + du_2}{\|cu_1 + du_2\|} \Big\|, \max_{i\in\{1,2\}} \inf_{a,b\in\mathbb{R}} \Big\| \frac{cu_1 + du_2}{\|cu_1 + du_2\|} - \frac{u_{i_n}}{\|u_{i_n}\|} \Big\|\Big) \to 0$$

as $n \to \infty$.

Proof.

$$\sup_{a\in\mathbb{R}} \inf_{c,d\in\mathbb{R}} \Big\| \frac{au_{1_n}}{\|au_{1_n}\|} - \frac{cu_1 + du_2}{\|cu_1 + du_2\|} \Big\| \le \sup_{a,b\in\mathbb{R}} \inf_{c,d\in\mathbb{R}} \Big\| \frac{au_{1_n} + bu_{2_n}}{\|au_{1_n} + bu_{2_n}\|} - \frac{cu_1 + du_2}{\|cu_1 + du_2\|} \Big\|$$

and

$$\sup_{b\in\mathbb{R}} \inf_{c,d\in\mathbb{R}} \Big\| \frac{bu_{2_n}}{\|bu_{2_n}\|} - \frac{cu_1 + du_2}{\|cu_1 + du_2\|} \Big\| \le \sup_{a,b\in\mathbb{R}} \inf_{c,d\in\mathbb{R}} \Big\| \frac{au_{1_n} + bu_{2_n}}{\|au_{1_n} + bu_{2_n}\|} - \frac{cu_1 + du_2}{\|cu_1 + du_2\|} \Big\|$$

Thus,

$$\sup_{i\in\{1,2\}} \inf_{c,d\in\mathbb{R}} \Big\| \frac{u_{i_n}}{\|u_{i_n}\|} - \frac{cu_1 + du_2}{\|cu_1 + du_2\|} \Big\| \le \sup_{a,b\in\mathbb{R}} \inf_{c,d\in\mathbb{R}} \Big\| \frac{au_{1_n} + bu_{2_n}}{\|au_{1_n} + bu_{2_n}\|} - \frac{cu_1 + du_2}{\|cu_1 + du_2\|} \Big\|$$

Using the same way, we get

$$\sup_{i\in\{1,2\}} \inf_{a,b\in\mathbb{R}} \Big\| \frac{u_i}{\|u_i\|} - \frac{au_{1_n} + bu_{2_n}}{\|au_{1_n} + bu_{2_n}\|} \Big\| \le \sup_{c,d\in\mathbb{R}} \inf_{a,b\in\mathbb{R}} \Big\| \frac{cu_1 + du_2}{\|cu_1 + du_2\|} - \frac{au_{1_n} + bu_{2_n}}{\|au_{1_n} + bu_{2_n}\|} \Big\|$$

Since $U_n \to U$ then

$$\max\Big(\max_{i\in\{1,2\}} \inf_{c,d\in\mathbb{R}} \Big\| \frac{u_{i_n}}{\|u_{i_n}\|} - \frac{cu_1 + du_2}{\|cu_1 + du_2\|} \Big\|, \max_{i\in\{1,2\}} \inf_{a,b\in\mathbb{R}} \Big\| \frac{cu_1 + du_2}{\|cu_1 + du_2\|} - \frac{u_{i_n}}{\|u_{i_n}\|} \Big\|\Big)$$

$$\le \max\Big(\sup_{a,b\in\mathbb{R}} \inf_{c,d\in\mathbb{R}} \Big\| \frac{au_{1_n} + bu_{2_n}}{\|au_{1_n} + bu_{2_n}\|} - \frac{cu_1 + du_2}{\|cu_1 + du_2\|} \Big\|, \sup_{c,d\in\mathbb{R}} \inf_{a,b\in\mathbb{R}} \Big\| \frac{cu_1 + du_2}{\|cu_1 + du_2\|} - \frac{au_{1_n} + bu_{2_n}}{\|au_{1_n} + bu_{2_n}\|} \Big\|\Big) \to 0$$

as $n \to \infty$. ∎

4. RELATIONSHIP TO ANGLE IN AN INNER PRODUCT SPACE

The notion of angles between two subspaces of a Euclidean space R^n has attracted many researchers since the 1950's (see, for instance, [2], [4], [5]). These angles are known to statisticians and numerical analysts as canonical or principal angles. Throughout this section, let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space of dimension 2 or higher (may be infinite), and $U = span\{u_1, u_2\}$ and $V = span\{v_1, v_2\}$ be 2-dimensional subspaces of X, where $\{u_1, u_2\}$ and $\{v_1, v_2\}$ are orthonormal. Then, as is suggested by Gunawan, Neswan and Setya-Budhi in [8], one can define the angle $\theta(U, V)$ be the angle between U and V, given by

$$\cos^2\theta(U, V) = \det\theta(M^T M)$$

where $M = [\langle u_i, v_k\rangle]^T$ being a 2×2 matrix. In [7], Gunawan and Neswan have got the following formula.

Theorem 4.1. (Fact 2 in [7])

$$\cos\theta(U, V) = \max_{\substack{u\in U\\ \|u\|=1}} \max_{\substack{v\in V\\ \|v\|=1}} \langle u, v\rangle \cdot \min_{\substack{u\in U\\ \|u\|=1}} \max_{\substack{v\in V\\ \|v\|=1}} \langle u, v\rangle.$$

In inner product spaces, the Definition 3.1 can be reformulated in the following theorem.

Theorem 4.2. *Let (U_n) is a sequence of 2-dimensional subspaces in X and U be a 2-dimensional subspace in X. A sequence (U_n) converges to U if and only if*

$$\max\left(\max_{u_n\in U_n}\min_{u\in U}\left\|\frac{u_n}{\|u_n\|} - \frac{u}{\|u\|}\right\|, \max_{u\in U}\min_{u_n\in U_n}\left\|\frac{u}{\|u\|} - \frac{u_n}{\|u_n\|}\right\|\right) \to 0$$

as $n \to \infty$.

Proof. In inner product spaces X, let $U = span\{u_1, u_2\}$ and $V = span\{v_1, v_2\}$ be 2-dimensional subspaces of X. Applying Theorem 6.4.1 in [3], for every $u \in U$, $\|u - v\|$ is minimized on $\{v \in U : \|v\| = 1\}$ at $v = \frac{\text{proj}_V u}{\|\text{proj}_V u\|} \in U$. Thus, we have

$$\inf_{v\in V}\left\|\frac{u}{\|u\|} - \frac{v}{\|v\|}\right\| = \min_{v\in V}\left\|\frac{u}{\|u\|} - \frac{v}{\|v\|}\right\|$$

Consequently,

$$\begin{aligned}\min_{v\in V}\left\|\frac{u}{\|u\|} - \frac{v}{\|v\|}\right\|^2 &= \min_{v\in V}\left\langle\frac{u}{\|u\|} - \frac{v}{\|v\|}, \frac{u}{\|u\|} - \frac{v}{\|v\|}\right\rangle = 2 - 2\max_{v\in V}\left\langle\frac{u}{\|u\|}, \frac{v}{\|v\|}\right\rangle \\ &= 2 - 2\max_{v\in V}\left\langle\frac{u}{\|u\|}, \frac{\text{proj}_V u}{\|\text{proj}_V u\|}\right\rangle = 2 - 2\|\text{proj}_V u\|.\end{aligned}$$

We know that $\text{proj}_V u = \sum_{k=1}^{2}\langle u, u_k\rangle u_k$ (orthogonal projection u to V).
Consider the function $f(u) = \|\text{proj}_V u\|$, $u \in U$, $\|u\| = 1$.
To find extreme values, we examine the function $g := f^2$ with

$$g(u) = \|\text{proj}_V u\|^2 = \sum_{k=1}^{2}\langle u, u_k\rangle^2, \; u \in U, \|u\| = 1.$$

If $u = (\cos t)u_1 + (\sin t)u_2$ then we can view g as a function of t on $[0, \pi]$, with

$$g(t) = \sum_{k=1}^{2}[(\cos t)\langle u_1, u_k\rangle + (\sin t)\langle u_2, u_k\rangle]^2$$

Expanding the series and using trigonometric identities, we can rewrite g as

$$g(t) = \frac{1}{2}[C + \sqrt{A^2 + B^2}\cos(2t - \alpha)]$$

where

$$A = \sum_{k=1}^{2}[\langle u_1, u_k\rangle^2 - \langle u_2, u_k\rangle^2], B = 2\sum_{k=1}^{2}[\langle u_1, u_k\rangle\langle u_2, u_k\rangle],$$

$$C = \sum_{k=1}^{2}[\langle u_1, u_k\rangle^2 + \langle u_2, u_k\rangle^2]$$

and $\tan\alpha = B/A$. From this expression, we see that g has the minimum value $n = \frac{C-\sqrt{A^2+B^2}}{2}$. Thus we get

$$\inf_{u\in U} \|\mathrm{proj}_V u\| = \min_{u\in U} \|\mathrm{proj}_V u\|$$

Consequently,

$$\max_{u\in U} \min_{v\in V} \left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\|^2 = 2 - 2 \min_{u\in U} \|\mathrm{proj}_V u\|$$

So, we have

$$\sup_{u\in U} \inf_{v\in V} \left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\| = \max_{u\in U} \min_{v\in V} \left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\|$$

By using the equation from Definition 2.1, we have

$$\sup_{u_n\in U_n} \inf_{u\in U} \left\| \frac{u_n}{\|u_n\|} - \frac{u}{\|u\|} \right\| = \max_{u_n\in U_n} \min_{u\in U} \left\| \frac{u_n}{\|u_n\|} - \frac{u}{\|u\|} \right\|$$

∎

Theorem 4.3. *Let U_n and U be 2-dimensional subspaces of a normed space X and $\theta(U_n, U)$ be an angle between U_n and U. If $\theta(U_n, U) \to 0$ then $\theta(U_n, U) \to 0$ as $n \to \infty$.*

Proof. We know that

$$\begin{aligned}\min_{u\in U} \left\| \frac{u_n}{\|u_n\|} - \frac{u}{\|u\|} \right\|^2 &= \min_{u\in U} \left\langle \frac{u_n}{\|u_n\|} - \frac{u}{\|u\|}, \frac{u_n}{\|u_n\|} - \frac{u}{\|u\|} \right\rangle = \min_{\substack{u\in U\\ \|u\|=1}} (2 - 2\langle u_n, u\rangle) \\ &= 2 - 2 \max_{\substack{u\in U\\ \|u\|=1}} \langle u_n, u\rangle\end{aligned}$$

Thus,

$$\max_{\substack{u\in U\\ \|u\|=1}} \langle u_n, u\rangle = \frac{1}{2}\left(2 - \min_{u\in U} \left\| \frac{u_n}{\|u_n\|} - \frac{u}{\|u\|} \right\|^2\right)$$

So, we have

$$\begin{aligned}\max_{\substack{u_n\in U_n\\ \|u_n\|=1}} \max_{\substack{u\in U\\ \|u\|=1}} \langle u_n, u\rangle &= \max_{\substack{u_n\in U_n\\ \|u_n\|=1}} \frac{1}{2}\left(2 - \min_{u\in U} \left\| \frac{u_n}{\|u_n\|} - \frac{u}{\|u\|} \right\|^2\right) \\ &= 1 - \frac{1}{2} \min_{u_n\in U_n} \min_{u\in U} \left\| \frac{u_n}{\|u_n\|} - \frac{u}{\|u\|} \right\|^2.\end{aligned}$$

According to Theorem 4.1, we get

$$\cos\theta(U_n, U) = \max_{\substack{u_n\in U_n\\ \|u_n\|=1}} \max_{\substack{u\in U\\ \|u\|=1}} \langle u_n, u\rangle \cdot \min_{\substack{u_n\in U_n\\ \|u_n\|=1}} \max_{\substack{u\in U\\ \|u\|=1}} \langle u_n, u\rangle.$$

Then we have

$$\begin{aligned}\cos\theta(U_n, U) &= \max_{\substack{u_n\in U_n\\ \|u_n\|=1}} \max_{\substack{u\in U\\ \|u\|=1}} \langle u_n, u\rangle \cdot \min_{\substack{u_n\in U_n\\ \|u_n\|=1}} \max_{\substack{u\in U\\ \|u\|=1}} \langle u_n, u\rangle \le \left(\max_{\substack{u_n\in U_n\\ \|u_n\|=1}} \max_{\substack{u\in U\\ \|u\|=1}} \langle u_n, u\rangle\right)^2 \\ &= \left[1 - 2 \min_{u_n\in U_n} \min_{u\in U} \left\| \frac{u_n}{\|u_n\|} - \frac{u}{\|u\|} \right\|^2\right]^2\end{aligned}$$

Since $U_n \to U$ as $n \to \infty$, then $\max_{u_n \in U_n} \min_{u \in U} \left\| \frac{u_n}{\|u_n\|} - \frac{u}{\|u\|} \right\| \to 0$ as $n \to \infty$.
We also have $\max_{u_n \in U_n} \min_{u \in U} \left\| \frac{u_n}{\|u_n\|} - \frac{u}{\|u\|} \right\| \to 0$. Thus,

$$\min_{u_n \in U_n} \min_{u \in U} \left\| \frac{u_n}{\|u_n\|} - \frac{u}{\|u\|} \right\|^2 \le \max_{u_n \in U_n} \min_{u \in U} \left\| \frac{u_n}{\|u_n\|} - \frac{u}{\|u\|} \right\|^2$$

Consequently,

$$\theta(U_n, U) \le \Big(\max_{\substack{u_n \in U_n \\ \|u_n\|=1}} \max_{\substack{u \in U \\ \|u\|=1}} \langle u_n, u \rangle \Big)^2 = \left[1 - 2 \min_{u_n \in U_n} \min_{u \in U} \left\| \frac{u_n}{\|u_n\|} - \frac{u}{\|u\|} \right\|^2 \right]^2 \to 1$$

So, $\theta(U_n, U) \to 0$ as $n \to \infty$. ∎

Theorem 4.4. *Let U_n and U be 2-dimensional subspaces of a normed space X and $\theta(U_n, U)$ be an angle between U_n and U. If $\theta(U_n, U) \to 0$ then $\theta(U_n, U) \to 0$ as $n \to \infty$.*

Proof. For each $u_n \in U_n$, $\langle u_n, u \rangle$ is maximized on $\{u \in U : \|u\| = 1\}$ at $u = \frac{\mathrm{proj}_U u_n}{\|\mathrm{proj}_U u_n\|}$ and

$$\max_{\substack{u \in U \\ \|u\|=1}} \langle u_n, u \rangle = \left\langle u_n, \frac{\mathrm{proj}_U u_n}{\|\mathrm{proj}_U u_n\|} \right\rangle = \|\mathrm{proj}_U u_n\| \le 1.$$

$$\begin{aligned}
\max_{u_n \in U_n} \min_{u \in U} \left\| \frac{u_n}{\|u_n\|} - \frac{u}{\|u\|} \right\|^2 &= \max_{\substack{u_n \in U_n \\ \|u_n\|=1}} \min_{\substack{u \in U \\ \|u\|=1}} \|u_n - u\|^2 \\
&= \max_{\substack{u_n \in U_n \\ \|u_n\|=1}} \min_{\substack{u \in U \\ \|u\|=1}} (2 - 2\langle u_n, u \rangle) \\
&= 2 - 2 \min_{\substack{u_n \in U_n \\ \|u_n\|=1}} \max_{\substack{u \in U \\ \|u\|=1}} \langle u_n, u \rangle \\
&\le \Big(2 - 2 \min_{\substack{u_n \in U_n \\ \|u_n\|=1}} \max_{\substack{u \in U \\ \|u\|=1}} \langle u_n, u \rangle \Big) \Big(\max_{\substack{u_n \in U_n \\ \|u_n\|=1}} \max_{\substack{u \in U \\ \|u\|=1}} \langle u_n, u \rangle \Big) \\
&= 2 \max_{\substack{u_n \in U_n \\ \|u_n\|=1}} \max_{\substack{u \in U \\ \|u\|=1}} \langle u_n, u \rangle - 2 \min_{\substack{u_n \in U_n \\ \|u_n\|=1}} \max_{\substack{u \in U \\ \|u\|=1}} \langle u_n, u \rangle \cdot \max_{\substack{u_n \in U_n \\ \|u_n\|=1}} \max_{\substack{u \in U \\ \|u\|=1}} \langle u_n, u \rangle \\
&\le 2 - 2 \min_{\substack{u_n \in U_n \\ \|u_n\|=1}} \max_{\substack{u \in U \\ \|u\|=1}} \langle u_n, u \rangle \cdot \max_{\substack{u_n \in U_n \\ \|u_n\|=1}} \max_{\substack{u \in U \\ \|u\|=1}} \langle u_n, u \rangle \\
&= 2 - 2 \cos \theta(U_n, U) \to 0 \ (\text{Since } \theta(U_n, U) \to 0 \, as \, n \to \infty)
\end{aligned}$$

and

$$
\begin{aligned}
\max_{u\in U}\min_{u_n\in U_n}\left\|\frac{u}{\|u\|}-\frac{u_n}{\|u_n\|}\right\|^2 &= \max_{\substack{u\in U\\ \|u\|=1}}\min_{\substack{u_n\in U_n\\ \|u_n\|=1}}\|u-u_n\|^2\\
&= \max_{\substack{u\in U\\ \|u\|=1}}\min_{\substack{u_n\in U_n\\ \|u_n\|=1}}(2-2\langle u,u_n\rangle)\\
&= 2-2\min_{\substack{u\in U\\ \|u\|=1}}\max_{\substack{u_n\in U_n\\ \|u_n\|=1}}\langle u,u_n\rangle\\
&\le \Big(2-2\min_{\substack{u\in U\\ \|u\|=1}}\max_{\substack{u_n\in U_n\\ \|u_n\|=1}}\langle u,u_n\rangle\Big)\Big(\max_{\substack{u\in U\\ \|u\|=1}}\max_{\substack{u_n\in U_n\\ \|u_n\|=1}}\langle u,u_n\rangle\Big)\\
&= 2\max_{\substack{u\in U\\ \|u\|=1}}\max_{\substack{u_n\in U_n\\ \|u_n\|=1}}\langle u,u_n\rangle-2\min_{\substack{u\in U\\ \|u\|=1}}\max_{\substack{u_n\in U_n\\ \|u_n\|=1}}\langle u,u_n\rangle\cdot\max_{\substack{u\in U\\ \|u\|=1}}\max_{\substack{u_n\in U_n\\ \|u_n\|=1}}\langle u,u_n\rangle\\
&\le 2-2\min_{\substack{u\in U\\ \|u\|=1}}\max_{\substack{u_n\in U_n\\ \|u_n\|=1}}\langle u,u_n\rangle\cdot\max_{\substack{u\in U\\ \|u\|=1}}\max_{\substack{u_n\in U_n\\ \|u_n\|=1}}\langle u,u_n\rangle\\
&= 2-2\cos\theta(U_n,U)\to 0\ (\text{Since}\,\theta(U,U_n)\to 0\,as\,n\to\infty).
\end{aligned}
$$

Thus, we have

$$
\max\Big(\max_{u_n\in U_n}\min_{u\in U}\left\|\frac{u_n}{\|u_n\|}-\frac{u}{\|u\|}\right\|,\max_{u\in U}\min_{u_n\in U_n}\left\|\frac{u}{\|u\|}-\frac{u_n}{\|u_n\|}\right\|\Big)\to 0
$$

as $n\to\infty$. So, $U_n\to U$ as $n\to\infty$. ∎

Remark 4.1. Theorem 4.3 and 4.4 show that, in any inner product spaces, $U_n\to U$ if and only if $\theta(U_n,U)\to 0$ as $n\to\infty$.

5. Concluding Remarks

Remark 4.1 can be utilised to prove the continuity of angle between two 2-dimensional subspaces, that is if $U_n\to U$ and $V_n\to V$ then

$$
\begin{aligned}
|\theta(U_n,V_n)-\theta(U,V)| &= |\theta(U_n,V_n)-\theta(U_n,V)|+|\theta(U_n,V)-\theta(U,V)|\\
&= \frac{\pi}{2}\theta(U_n,U)+\frac{\pi}{2}\theta(U_n,U)\to 0.
\end{aligned}
$$

In other words, the continuity satisfies.

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